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SOME RECENT RESULTS ON THE THEORY OF TURBULENT FLOW OF GASES OF HIGHLY VARIABLE DENSITY

Witold Szablewski

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16. Abstract

Some theoretical results are obtained on turbulent flow in gases of variable density. The cases considered include turbulent boundary-layer flow and turublent expansion of jets in a surrounding medium. In the latter case, a "mixing wedge" is obtained which swings toward the hotter jet as the temperature difference between the jet and the surrounding air increases. For boundary-layer flow, a new relation between temperature and velocity (Equation 50) is derived and confirmed by experiment.

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SOME RECENT RESULTS ON THE THEORY OF TURBULENT FLOW OF GASES OF HIGHLY VARIABLE DENSITY

Witold Szablewski

I. Introduction /5*

Some phenomena, fundamentally important for engineering, in the mechanics of turbulent flow are connected with large variations in density. One topic presently of great interest is the question of flow processes over the wings of high-speed aircraft and over the bodies of rockets, in which the wall undergoes vigorous heating, resulting in large density differences in the turbulent boundary layer clinging to these objects. The same situation can be found in high-speed jet engines. Another topic which is very important technically is the turbulent expansion of jets of widely varying density in the surrounding medium, such as the expansion of a natural-gas jet in air; processes of this type are also of great importance in the study of combustion processes.

In the following, we will report on some theoretical results relative to these phenomena, results obtained at the Research Institute for Applied Mathematics and Mechanics of the German Academy of Sciences. Our results pay particular attention -- as will be discussed further -- to the <u>influence of density fluctuations.</u>

Our flow models are restricted to turbulent flows which, averaged over time, are steady, and have either plane or rotation-symmetric character. Moreover, these flows are characterized by having a principal flow direction: for planar flows we will designate this direction with x and the perpendicular direction with y, and the corresponding velocity components with u and v,

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*Numbers in the margin indicate pagination in the foreign text.

so that $|v| \lll |u|$. In other words, these are flows which are at a small angle to the x-axis.

These restricted cases include the most important boundary-layer flows over profiles as well as jets.

II. Free-Turbulence Flows of Gases with Highly Variable Densities

By free turbulence is meant turbulent flows without limiting walls, such as jets or wakes.

By comparison with boundary-layer flows (along limiting walls), these flows are of a much simpler nature, to the extent that molecular effects appear negligible in the determination of the mean state parameters of the mixing field. Moreover, the assumption of constant pressure (p = const) has worked well for flows in the subsonic region, to which we restrict ourselves.

A. Equations

For two-dimensional flows which are steady when averaged over time, one obtains the system of equations:

continuity:

$$\frac{\partial \overline{\varrho u}}{\partial x} + \frac{\partial \overline{\varrho v}}{\partial y} = 0 , \tag{1}$$

momentum: /6

$$\frac{\partial \overline{\varrho u^2}}{\partial x} + \frac{\partial \overline{\varrho u v}}{\partial y} = 0 , \tag{2}$$

energy:

$$\frac{\partial \overline{\varrho c_p T u}}{\partial x} + \frac{\partial \overline{\varrho c_p T v}}{\partial y} = 0 , \tag{3}$$

concentration:

$$\frac{\partial \overline{\varrho c u}}{\partial x} + \frac{\partial \overline{\varrho c v}}{\partial y} = 0 . \tag{4}$$

(Lines over the symbols indicate mean values over time.)

Here ρ stands for the density, T for the absolute temperature, c_p for the specific heat at constant pressure, and c for the concentration of a chemically different gas in the mixing field.

In order to obtain equations for the mean state parameters, the instantaneous state parameters are split, by the method of O. Reynolds [1] into their mean values and the instantaneous fluctuations

$$u = \bar{u} + u', \qquad v = \bar{v} + v',$$
$$T = \bar{T} + T', \qquad \varrho = \bar{\varrho} + \varrho', \qquad c = \bar{c} + c'.$$

For instance, the continuity equation then becomes

$$\frac{\partial(\bar{\varrho}\,\bar{u} + \overline{\varrho' u'})}{\partial x} + \frac{\partial(\bar{\varrho}\,\bar{v} + \overline{\varrho' v'})}{\partial y} = 0\,. \tag{5}$$

The main principle in the further theoretical treatment is the experimental discovery that for all state parameters with the exception of the transverse component v of the flow, the instantaneous fluctuations are small in comparison with the mean values. In the flow models considered here, however, with the main flow direction x, the instantaneous fluctuations v' assume large values in comparison with the small transverse component v [2].

Accordingly, in the continuity equation (5), we can neglect $\overline{\rho' u'}$ in comparison with $\bar{\rho}\bar{u}$, but not $\overline{\rho' v'}$ in comparison with $\bar{\rho}\,\bar{v}$; we then obtain

continuity:

$$\frac{\partial \bar{\varrho}\,\bar{u}}{\partial x} + \frac{\partial \bar{\varrho}\,\bar{v}}{\partial y} = \partial/\partial y\left(-\overline{\varrho' v'}\right). \tag{6}$$

In an analogous fashion, with

$$\overline{\varrho u v} \approx \bar{\varrho}\,\bar{u}\,\bar{v} + \bar{\varrho}\,\overline{u' v'} + \bar{u}\,\overline{\varrho' v'},$$
$$\overline{\varrho c_p T v} \approx \bar{\varrho}\,\overline{c_p T}\,\bar{v} + \bar{\varrho}\,\overline{(c_p T)' v'} + \overline{c_p T}\,\overline{\varrho' v'},$$
$$\overline{\varrho c v} \approx \bar{\varrho}\,\bar{c}\,\bar{v} + \bar{\varrho}\,\overline{c' v'} + \bar{c}\,\overline{\varrho' v'}$$

we obtain the equations

momentum:

$$\frac{\partial \bar{\varrho}\,\bar{u}^2}{\partial x} + \frac{\partial \bar{\varrho}\,\bar{u}\,\bar{v}}{\partial y} = \partial/\partial y \left\{ -\bar{\varrho}\,\overline{u' v'} - \bar{u}\,\overline{\varrho' v'} \right\}, \qquad (7)$$

energy:

$$\frac{\partial \bar{\varrho}\,\overline{c_p T}\,\bar{u}}{\partial x} + \frac{\partial \bar{\varrho}\,\overline{c_p T}\,\bar{v}}{\partial y} = \partial/\partial y \left\{ -\bar{\varrho}\,\overline{(c_p T)' v'} - \overline{c_p T}\,\overline{\varrho' v'} \right\}, \qquad (8)$$

concentration: /7

$$\frac{\partial \bar{\varrho}\,\bar{c}\,\bar{u}}{\partial x} + \frac{\partial \bar{\varrho}\,\bar{c}\,\bar{v}}{\partial y} = \partial/\partial y \left\{ -\bar{\varrho}\,\overline{c' v'} - \bar{c}\,\overline{\varrho' v'} \right\}. \qquad (9)$$

Hence, from the standpoint of the mean state parameters, the mechanism of turbulent mixing generates fluxes of mass, momentum, enthalpy, and concentration. Intuitively, it can be imagined that -- by analogy with the ideas of kinetic gas theory, which assumes the transfer of fluid properties from one layer to the next by means of molecules -- the transport in this case is supplied by fluid elements.

Equations (6-9) [3] can be viewed as experimentally well-established foundations. In order to use them to obtain information on the field of mean state parameters, however, the correlations appearing in the equations for the fluctuations must be described in functional form using the mean state parameters. At the current state of the theory, one must rely on hypotheses.

One hypothesis which has always proved workable for phenomena of free turbulence and which is due to L. Prandtl [4] states that in any cross section of the flow (section perpendicular to the principal direction of flow[1]), correlations of the form $\overline{m'v'}$ are proportional to the gradients of the mean state parameter $\bar{m}(y)$.

For the correlations of Eqs. (6-9), this gives

$$\left.\begin{aligned} -\overline{u'\,v'} &= \varepsilon(x)\,\frac{\partial \bar{u}}{\partial y}\,, \\ -\overline{(c_p\,T)'\,v'} &= E\,\varepsilon(x)\,\frac{\partial \overline{c_p\,T}}{\partial y}\,, \\ -\overline{\varrho'\,v'} &= E\,\varepsilon(x)\,\frac{\partial \bar{\varrho}}{\partial y}\,, \\ -\overline{c'\,v'} &= E\,\varepsilon(x)\,\frac{\partial \bar{c}}{\partial y}\,. \end{aligned}\right\} \tag{10}$$

The coefficient E allows for the circumstance that -- as experiments have shown -- in transport from one layer to another, the "mixing path" for the substantial properties is different from that for the velocity which is exposed to pressure fluctuations during transport (by the same carrier). According to numerous measurements [1], the coefficient for rotation-symmetric jets has the value

$$E = 2 \tag{11}$$

The factor $\varepsilon(x)$ appearing in (10) has the dimensions of a kinematic viscosity. The factors available for determining it are the width of the velocity field $b_1(x)$ and the velocity range $|\bar{u}_{max} - \bar{u}_{min}|$. Dimensional analysis then shows that

$$\varepsilon(x) = \varkappa_1\, b_1(x)\, |\bar{u}_{max} - \bar{u}_{min}|\,, \tag{12}$$

[1] Cf. e.g. L. Prandtl, Guide to fluid dynamics [Führer durch die Strömungslehre], Braunschweig, 1949.

where x_1 is an empirical coefficient of the theory.

The calculations will be based on the following system of equations[2]:

continuity:

$$\frac{\partial\bar{\varrho}\,\bar{u}}{\partial x}+\frac{\partial\bar{\varrho}\,\bar{v}}{\partial y}=E\,\varepsilon(x)\,\frac{\partial^2\bar{\varrho}}{\partial y^2}, \tag{13}$$

momentum:

$$\frac{\partial\bar{\varrho}\,\bar{u}^2}{\partial x}+\frac{\partial\bar{\varrho}\,\bar{u}\,\bar{v}}{\partial y}=\varepsilon(x)\,\partial/\partial y\left\{\bar{\varrho}\,\frac{\partial\bar{u}}{\partial y}+E\,\bar{u}\,\frac{\partial\bar{\varrho}}{\partial y}\right\}, \tag{14}$$

~~ener~~ energy: /8

$$\frac{\partial\bar{\varrho}\,\overline{c_p\,T'}\,\bar{u}}{\partial x}+\frac{\partial\bar{\varrho}\,\overline{c_p\,T'}\,\bar{v}}{\partial y}=E\,\varepsilon(x)\,\frac{\partial^2\bar{\varrho}\,\overline{c_p\,T}}{\partial y}, \tag{15}$$

concentration:

$$\frac{\partial\bar{\varrho}\,\bar{c}\,\bar{u}}{\partial x}+\frac{\partial\bar{\varrho}\,\bar{c}\,\bar{v}}{\partial y}=E\,\varepsilon(x)\,\frac{\partial^2\bar{\varrho}\,\bar{c}}{\partial y^2}. \tag{16}$$

We make special mention of the fact that this system of equations is distinguished from other systems [5] by taking density fluctuations into account.

B. Turbulent Expansion of Hot-Air Jets in Stationary or Moving Surrounding Air

In the given form, the system of equations (13-16) can assist in the calculation of effects of free turbulence of gases with highly variable densities, under the restriction mentioned at the outset. In rotation-symmetric flows, we must employ the corresponding formulations of Eqs. (13-16) in cylindrical coordinates.

On this basis, we have in several works described theoretically the turbulent expansion of round hot-air jets in moving and

[2] W. Szablewski, op. cit.

stationary surrounding air, for which there are ample measurements. For instance, the jets from the engines of high-speed aircraft fall under this heading, but it also includes problems of cooling by means of injecting cold-air jets in surrounding hot air.

We already observe that the problem of turbulent expansion of jets of varying chemical density in air (e.g. the turbulent expansion of a natural-gas jet in air) leads to the same formalism from the mathematical point of view.

The mixing field of a hot-air jet escaping from a round nozzle can be depicted schematically as in Fig. 1:

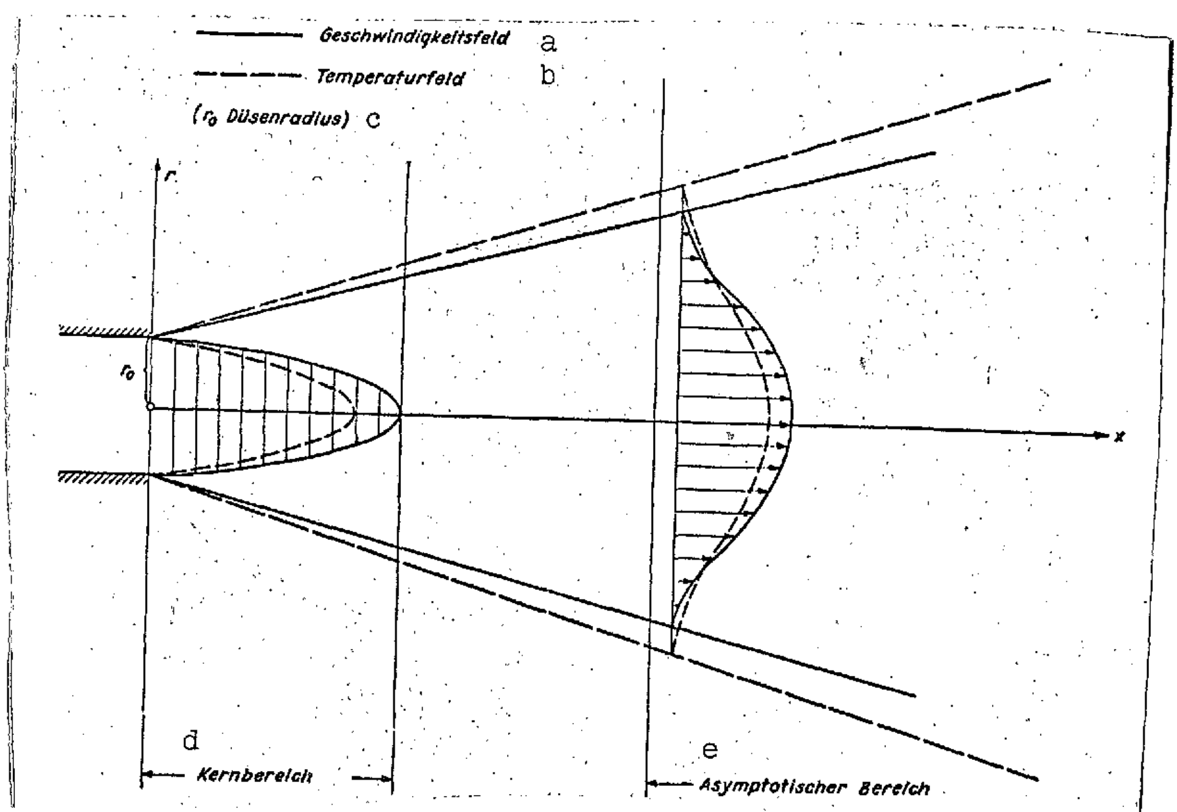

Fig. 1.

Key: a. Velocity field
b. Temperature field
c. Nozzle radius
d. Core region
e. Asymptotic region

The core region is characterized by the core of the jet, which has not yet been affected by intermixing. Further away from the mouth of the nozzle, we have the asymptotic region, in which the flow processes take place with geometrical and mechanical similarity. /9

In conformity with the fact that there is a mixing path for the transport of substantial properties such as enthalpy which is different from that for the velocity -- cf. Eq. (11) -- the velocity and temperature fields exhibit different structures.

1. Turbulent Mixing of Two-Dimensional Hot-Air Jets

From the theory of the jet, we first demonstrate the characteristic temperature effect or the effect of a large density difference between the jet and the surrounding air, using the following model:

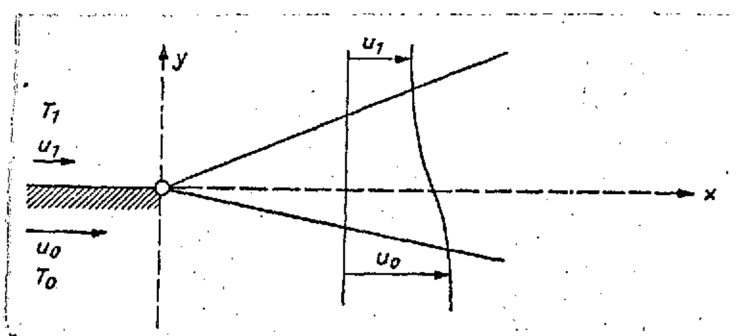

Fig. 2.

In the immediate vicinity of the edge of the nozzle, the flow field can be depicted schematically as in Fig. 2, if frictional effects are neglected. In other words, two air jets of different velocities and temperatures first separated by a partition, would then be mixed in an angular region.

In the cross sections perpendicular to the jet axis, the flow field can be viewed as geometrically and mechanically similar; the similarity coordinate is y/x. With the normalized factors

$$\varphi = \frac{\bar{u} - u_1}{u_0 - u_1}, \qquad \chi = \bar{\vartheta}/\vartheta_0 \,(\bar{\vartheta} = \bar{T} - T_1, \;\; \vartheta_0 = T_0 - T_1), \qquad \psi = \sigma\,\bar{v}/u_0$$

and with the variable $\eta = \sigma\, y/x$ containing the expansion factor

$$\sigma = \frac{1}{\sqrt{2\,\varkappa_1 c_1 \frac{u_0 - u_1}{u_0}}} \tag{17}$$

($b_1 = c_1 x$ = mixing width of velocity field)
and substituting $\bar{T}$ for $\bar{\rho}$ in accordance with the gas equation

$$\bar{\varrho}\,\bar{T} = \text{const} \tag{18}$$

derived from the assumption of constant pressure (cf. beginning of Section II), the system of equations (13-15) reduces to a system of the following differential equations [6]:

continuity:

$$-\frac{u_0 - u_1}{u_0}\,\eta\,\frac{d\varphi}{d\eta} + \frac{d\psi}{d\eta} = 0\,, \tag{19}$$

momentum:

$$\frac{d^2\varphi}{d\eta^2} + \frac{d\varphi}{d\eta}\left\{2\left[\left(\frac{u_0 - u_1}{u_0}\varphi + \frac{u_1}{u_0}\right)\varphi - \psi\right] - \underline{\frac{E+1}{1 + (\vartheta_0/T_1)\,\chi}\,\frac{d(\vartheta_0/T_1)\,\chi}{d\eta}}\right\} = 0\,; \tag{20}$$

energy:

$$\frac{d^2\chi}{d\eta^2} + \frac{d\chi}{d\eta}\left\{\frac{2}{E}\left[\cdots\right] - \underline{\frac{2}{1 + (\vartheta_0/T_1)\,\chi}\,\frac{d(\vartheta_0/T_1)\,\chi}{d\eta}}\right\} = 0\,. \tag{21}$$

The underlined terms are those which would not appear if the /10 density were constant.

Equation (19) is identical with

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} = 0$$

and expresses the fact that the mixing is isochoric (volume-preserving) under the assumption of constant pressure and neglecting molecular effects.

The boundary conditions are

$$\left.\begin{array}{l|l} \varphi, \chi \to \begin{matrix}1\\0\end{matrix} & \text{for} \quad \eta \to \begin{matrix}-\infty,\\ \infty\end{matrix} \\ \psi \to 0 & \text{for} \quad \eta \to -\infty . \end{array}\right\} \tag{22}$$

The last condition says that the transverse component $\bar{v}$ will vanish in the core of the jet.

For the special case $(u_0 - u_1)/u_0 \approx 0$, i.e. for jets of almost identical velocities and otherwise arbitrary temperature differences, the solution can be given explicitly as

$$\left.\begin{array}{l} \varphi = \gamma \int\limits_0^{\eta/\sqrt{E}} \frac{e^{-\eta^2}}{[\alpha\, \Phi(\eta/\sqrt{E}) + \beta]^{\beta+1}}\, d\eta + \delta , \\ \chi = \frac{1}{\vartheta_0/T_1} \left\{ \frac{1}{\alpha\, \Phi(\eta/\sqrt{E}) + \beta} - 1 \right\}. \end{array}\right.$$

with the constants

$$\alpha = \frac{1}{2}\, \frac{\vartheta_0/T_1}{1 + (\vartheta_0/T_1)}, \qquad \beta = \frac{1}{2}\left(1 + \frac{1}{1 + (\vartheta_0/T_1)}\right) \tag{23}$$

and

$$\gamma = \frac{-1}{\int\limits_{-\infty}^{\infty}}, \qquad \delta = \frac{\int\limits_0^{\infty}}{\int\limits_{-\infty}^{\infty}} .$$

$\Phi(y)$ is the error integral function

$$\Phi(y) = \frac{2}{\sqrt{\pi}} \int\limits_0^y e^{-y^2}\, dy .$$

The results are shown by Figs. 3, 4, and 5[3], which can be /11
interpreted to show that without any appreciable change in the generating angle, the wedge in which mixing occurs swings toward the hotter jet as the temperature difference increases.

[3]In Figs. 3, 4, and 5, $\sigma_* = E^{-1/2}\sigma$, and in Figs. 3 and 4, $\sigma_*\eta = E^{-1/2}\sigma y/x$.

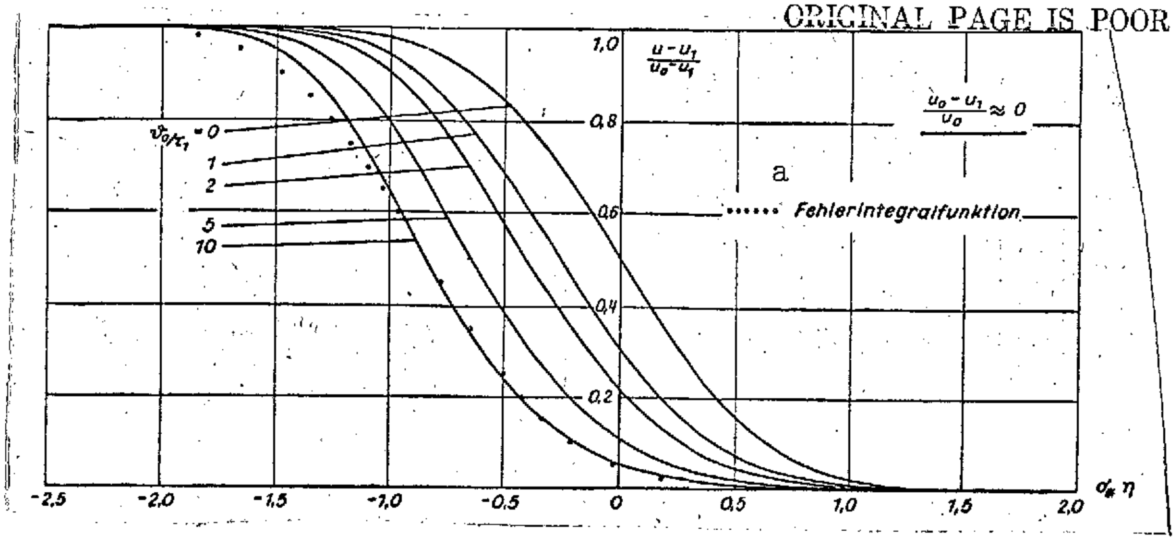

Fig. 3. Distribution of the longitudinal component of the velocity.

Key: a. Error integral function

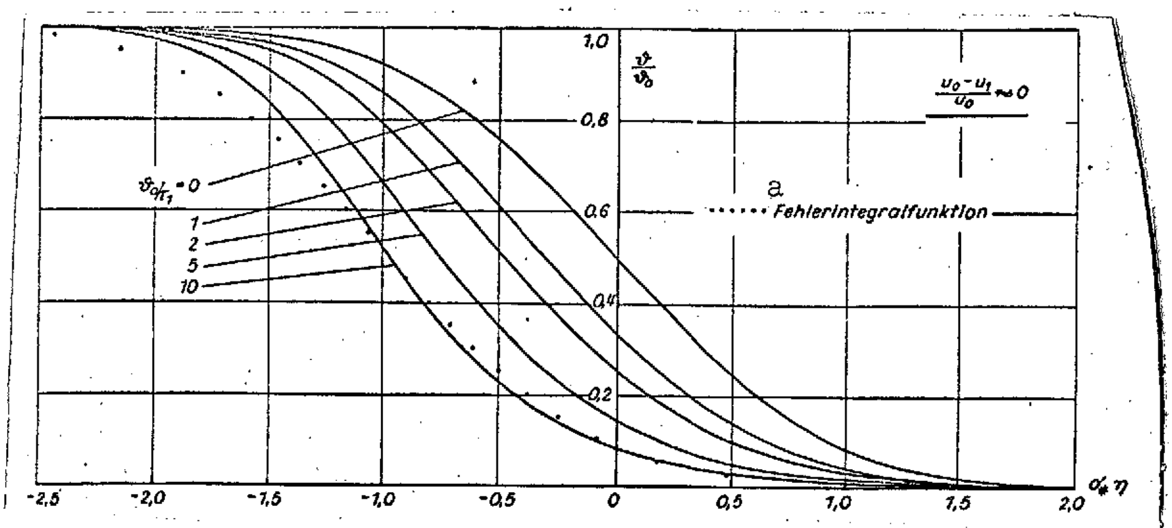

Fig. 4. Temperature distribution.

Key: a. Error integral function.

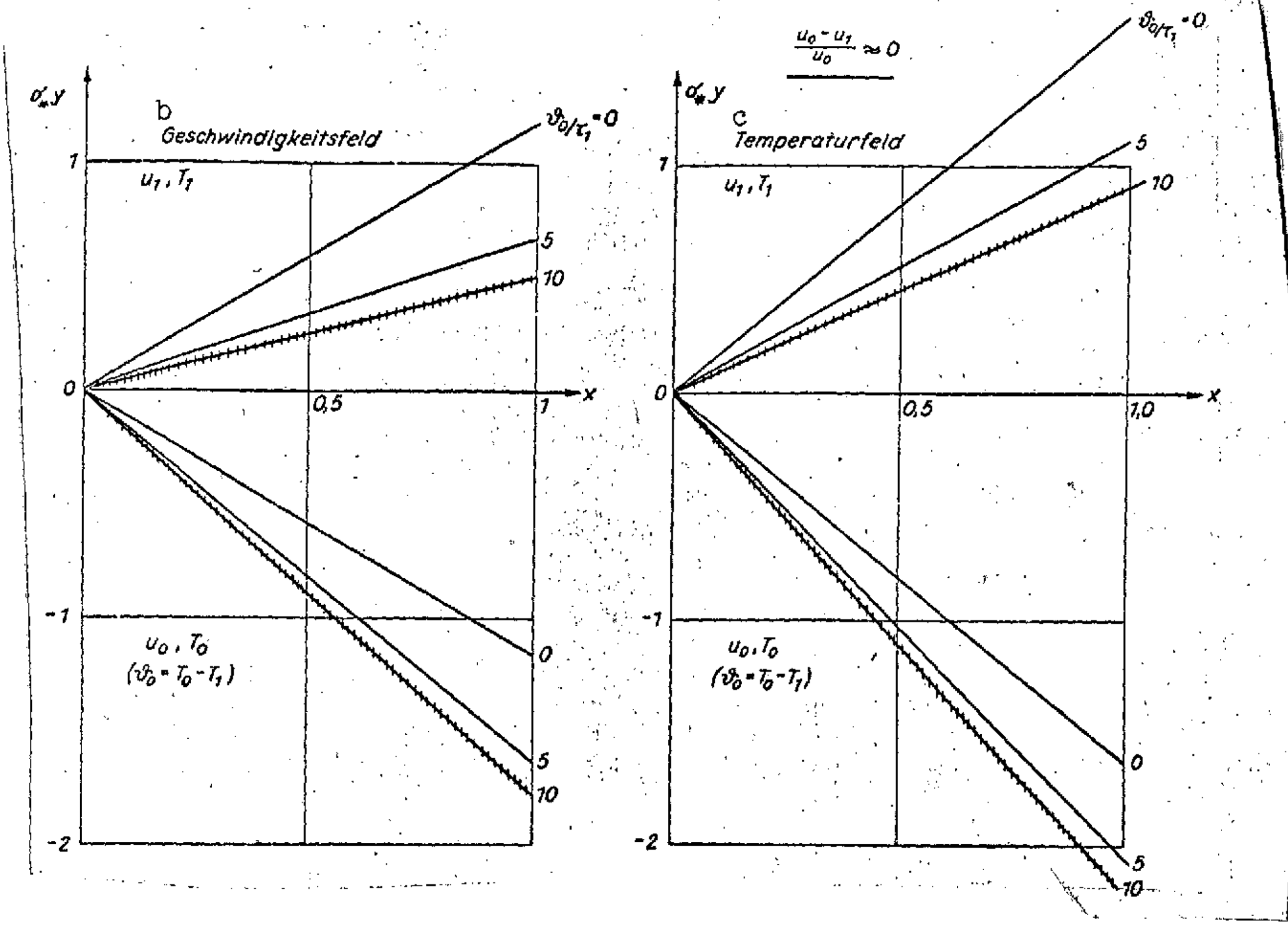

Fig. 5. Wedge of mixing zones.

Key: b. Velocity field
c. Temperature field

In the general case $(u_0 - u_1)/u_0 \neq 0$, the equations can be solved only numerically [7]; the solution can be obtained by using the iteration formulas /12

$$\left.\begin{aligned}
\varphi &= C_1 \int_0^\eta \left(1 + \frac{\vartheta_0}{T_1}\chi\right)^{E+1} \exp\left\{-2\int_0^\eta \left[\left(\frac{u_0 - u_1}{u_0}\varphi + \frac{u_1}{u_0}\right)\eta - \psi\right] d\eta\right\} d\eta + C_2, \\
\frac{1}{1 + (\vartheta_0/T_1)\chi} &= D_1 \int_0^\eta \exp\left\{\frac{-2}{E}\int_0^\eta \left[\cdots\right] d\eta\right\} d\eta + D_2, \\
\psi &= \frac{u_0 - u_1}{u_0} \int_{-\infty}^\eta \eta \frac{d\varphi}{d\eta}\, d\eta
\end{aligned}\right\} \quad (24)$$

The integration constants C_1, C_2, D_1, and D_2 are to be determined in accordance with the conditions in (22). The resulting distributions again show the previously mentioned feature.

If the width σc_1 of the calculated velocity distributions is read off between the limits 0.95 and 0.05, a comparison with measurements for the empirical coefficient x_1 (cf. Eq. (12) with (17)) yields the value $x_1 = 0.0082$.

By comparing the theory with experiments[4], we obtain Fig. 6, which, in particular, shows the temperature effect. Practically speaking, both measured jets opened into stationary air; while the temperature of one jet was the same as that of the surrounding air, the temperature of the other jet was 440°C higher. The quasi-parallel displacement of the curves in the diagram shows the rotation of the mixing wedge toward the warmer side.

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[4]It should be remarked that the theory given here -- as mentioned at the outset in II, B 1,-- ignores the effects of friction at the walls of the nozzle, and assumes rectangular velocity profiles in the plane of the nozzle aperture. In applications to practical problems, the momentum loss relative to the theoretical model can, if necessary, be allowed for by following a suggestion of O. Pabst and introducing an effective nozzle radius r_0^* as defined by the equation

$$\int_0^\infty \varrho u (u - u_1) r \, dr = \varrho_0 u_0 (u_0 - u_1) \frac{r_0^{*2}}{2}$$

In Fig. 6, such a correction was made on the measurements of O. Pabst; $\eta^* = (r - r_0^*)/x$. Without this correction, the measurements would indicate an even greater temperature effect. Accordingly, the points on the experimental isotachs in Fig. 8 are plotted with the coordinates $x/2r_0^*$ and $r/2r_0^*$.

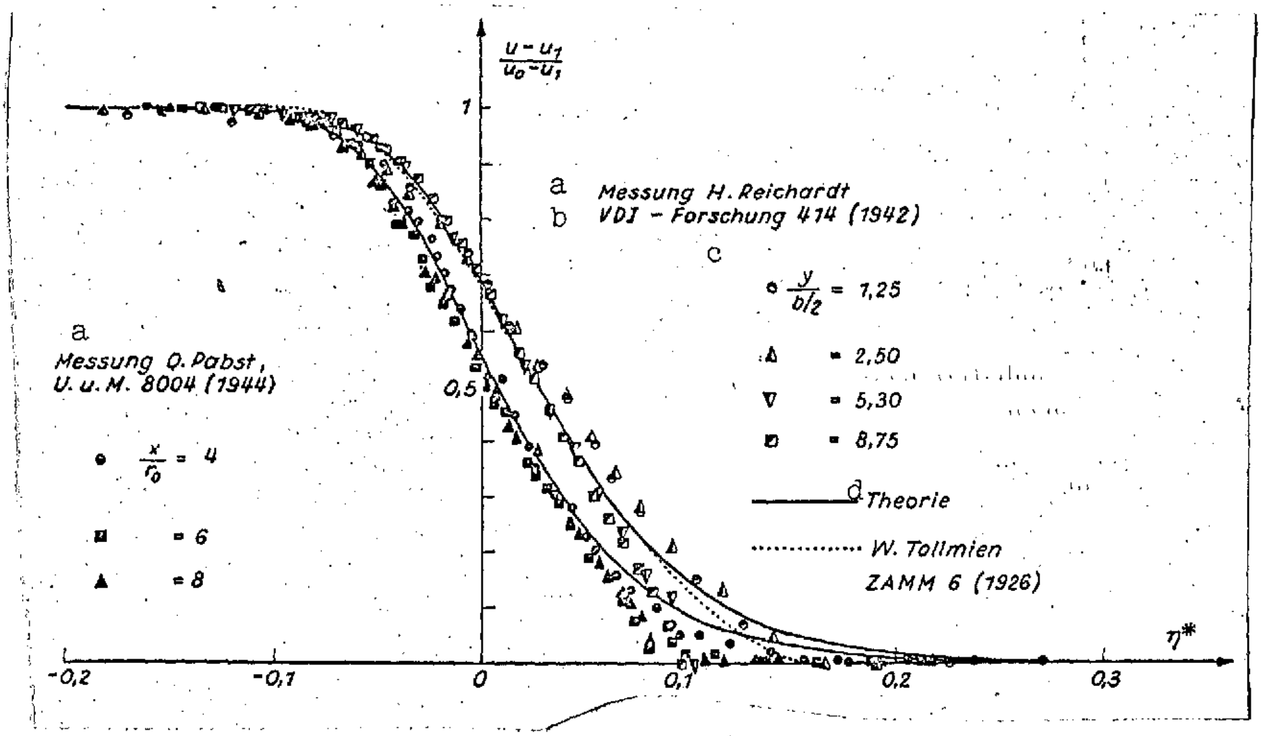

Fig. 6.

Key: a. Measurement
b. Association of German Engineers
c. Research
d. Theory

2. Core Region

Starting from the above model of turbulent mixing of two planar air jets, the mixing field of the core region (Fig. 1) can be calculated in the following fashion [8]:

Transforming the rotation-symmetric system of equations (13-15) for Fig. 1 to the jet coordinate

$$\frac{r - r_0}{x}, \tag{25}$$

we obtain, for constant σ -- this assumption can be considered a good approximation[5] -- in the variables /13

[5] From the physical standpoint, the above assumption implies no loss of generality, since, according to a statement of W. Kauschus [Ing. Arch. 31 (1962)], the case of a general function σ(x) in the core region can be reduced to the case σ(x) = const by means of a transformation.

$$\xi = \frac{1}{\sigma}\frac{x}{r_0} \quad \text{and} \quad \eta = \sigma\frac{r - r_0}{x}$$

the system of partial differential equations:

continuity:

$$\frac{u_0 - u_1}{u_0}\left(\xi\frac{\partial\varphi}{\partial\xi} - \eta\frac{\partial\varphi}{\partial\eta}\right) + \frac{\partial\psi}{\partial\eta} + \psi\frac{\xi}{1 + \xi\eta} = 0\,, \tag{26}$$

momentum:

$$\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial\varphi}{\partial\eta}\left\{2\left[\left(\frac{u_0 - u_1}{u_0}\varphi + \frac{u_1}{u_0}\right)\eta - \psi\right] \underline{- (E + 1)\frac{(\vartheta_0/T_1)\,\partial\chi/\partial\eta}{1 + (\vartheta_0/T_1)\,\chi}} + \frac{\xi}{1 + \xi\eta}\right\}$$
$$- 2\left(\frac{u_0 - u_1}{u_0}\varphi + \frac{u_1}{u_0}\right)\xi\frac{\partial\varphi}{\partial\xi} = 0\,, \tag{27}$$

energy:

$$\frac{\partial^2\chi}{\partial\eta^2} + \frac{\partial\chi}{\partial\eta}\left\{\frac{2}{E}\left[\left(\frac{u_0 - u_1}{u_0}\varphi + \frac{u_1}{u_0}\right)\eta - \psi\right] \underline{- 2\frac{(\vartheta_0/T_1)\,\partial\chi/\partial\eta}{1 + (\vartheta_0/T_1)\,\chi}} + \frac{\xi}{1 + \xi\eta}\right\}$$
$$- \frac{2}{E}\left(\frac{u_0 - u_1}{u_0}\varphi + \frac{u_1}{u_0}\right)\xi\frac{\partial\chi}{\partial\xi} = 0\,. \tag{28}$$

The underlined terms are again those which are absent in a constant-density field.

The boundary conditions read /14

$$\begin{array}{lll} \varphi, \chi \to \begin{matrix}1\\0\end{matrix} & \text{for} & \eta \to \begin{matrix}-1/\xi\\ \infty\end{matrix} \\ \psi \to 0 & \text{for} & \eta \to -\dfrac{1}{\xi} \end{array} \tag{29}$$

(the jet axis has the jet coordinate $\eta = -1/\xi$).

For $\xi = 0$, the system (26-28) reduces to the equations (19-21), which have already been discussed (Section I). Thus the model of turbulent mixing of planar hot-air jets gives us the initial profiles.

The system of partial differential equations with the given boundary conditions can be solved numerically by means of a continuation method, in which the mixing field is built up in approximation and differential increments starting from the initial profiles. Characterizing the method, the first step requires the solution of a system of coupled ordinary differential equations of second order, and this solution can be obtained by means of iteration. The next steps then require only quadratures, but the amount of computation involved increases considerably with each further step.

Since the isotachs and isotherms within the core region are only slightly curved (cf. Fig. 7), two or three differential steps are sufficient with the formulation of the problem given here (transformation to the jet coordinate).

From the results obtained, which apply to both stationary and moving surrounding air, we display in Fig. 7 the field of isotachs and isotherms for the two cases /15

$(u_0 - u_1)/u_0 = 1$ (in stationary air); $\theta_0/T_1 = 0$ and 2. As a comparison shows, the effect of temperature is to shorten the mixing field longitudinally and laterally; this shortening corresponds to the swinging of the mixing wedge (Section I) toward the jet axis with the vertex at the edge of the nozzle. It can also be seen from Fig. 7 that the temperature field has a structure, with its greater width and lesser depth, which is considerably different from that of the velocity field -- an effect caused by the transmission ratio E = 2 (cf. Eq. 11).

From the comparison with measurements, we obtain Figs. 8 and 9. The measurements[6] reproducted in Fig. 8 refer to a hot-air jet escaping from a round nozzle with a velocity of /16

[6]See Footnote 4.

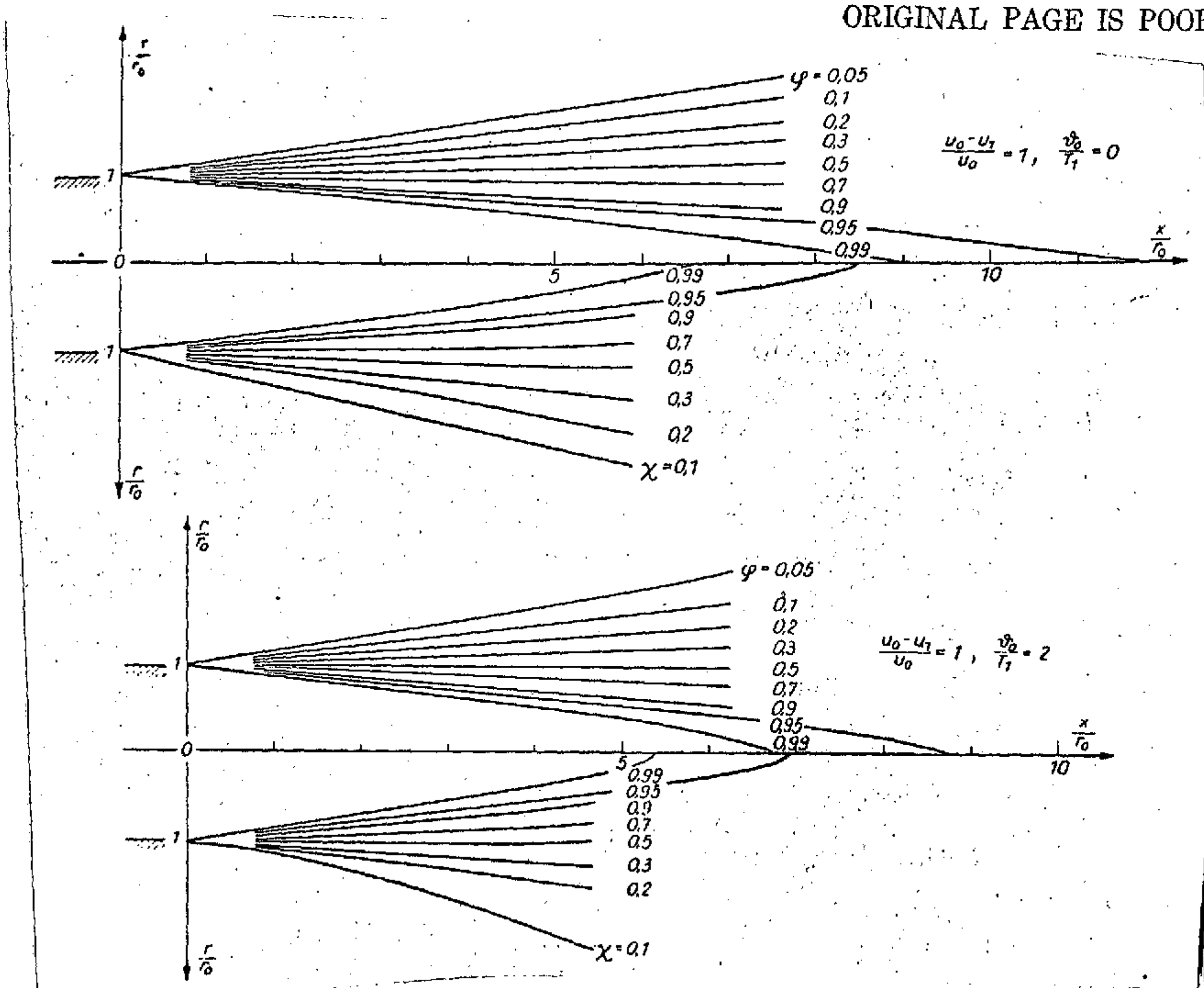

Fig. 7. Field of isotachs and isotherms.

400 m/sec and a temperature of 400°C, the jet mixing with an air stream of speed 100 m/sec and normal temperature. Figure 9 shows measurements of velocity and temperature along the jet axis, the jets being hot air at temperatures of 15° and 296°C above that of the surrounding stationary air. There is good agreement between theory and experiment.

We also mention that the empirical coefficient was already found to be x_1 = 0.0082 in Section 1.

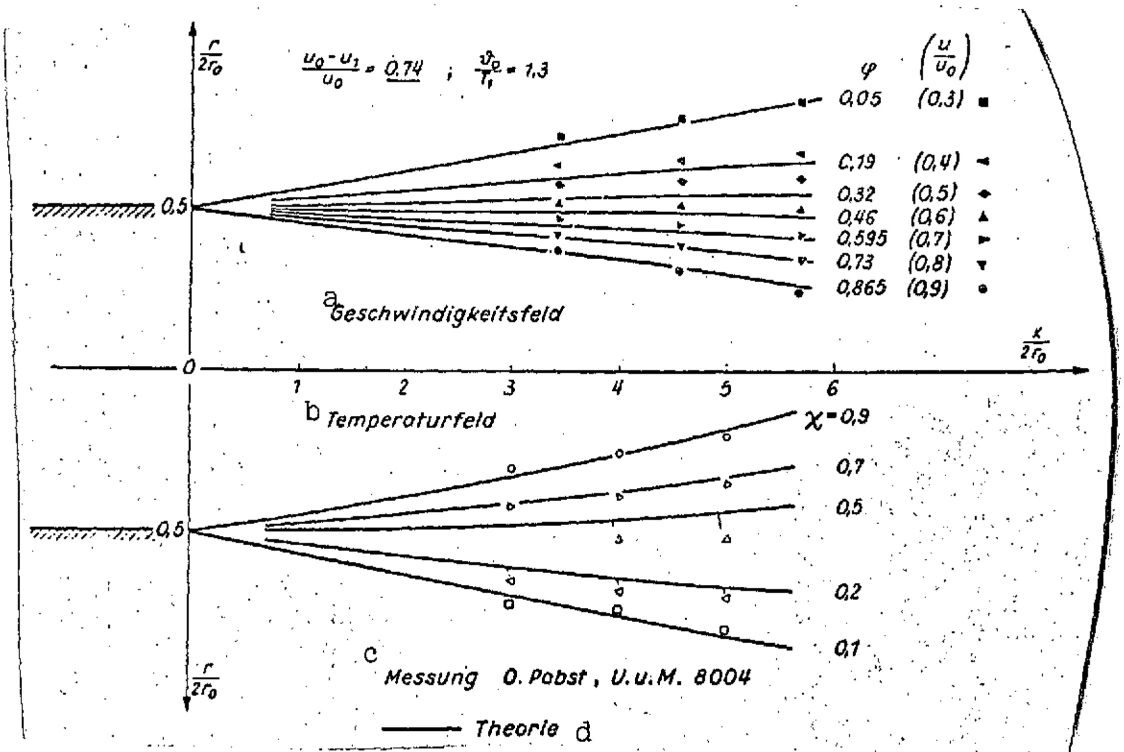

Fig. 8.

Key: a. Velocity field
b. Temperature field
c. Measurement
d. Theory

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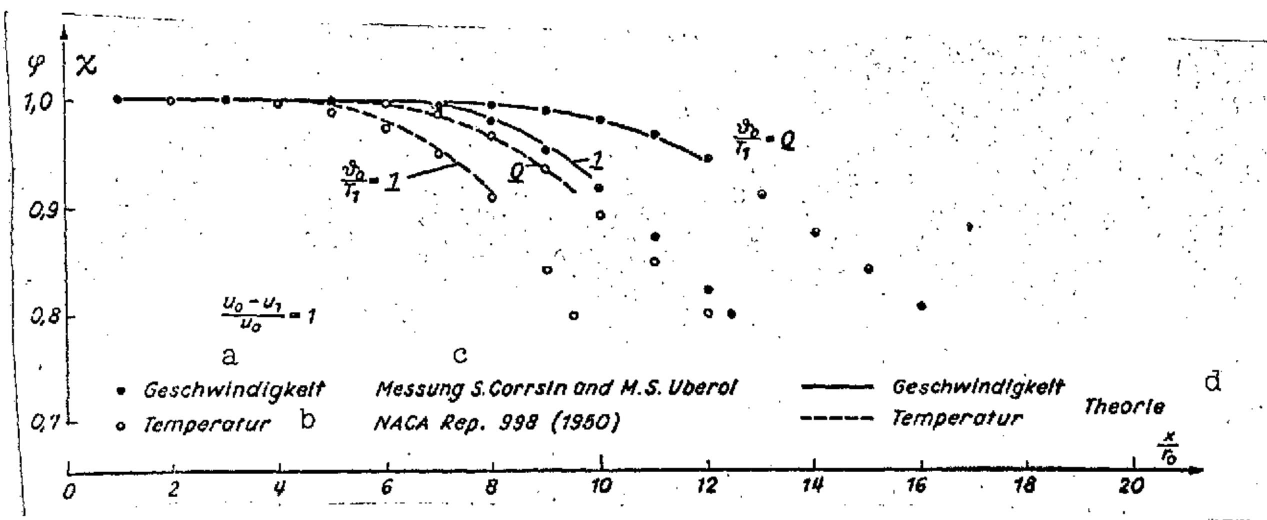

Fig. 9. Velocity and temperature along the jet axis.

Key: a. Velocity
b. Temperature
c. Measurement
d. Theory

3. Asymptotic Region

For the asymptotic region (Fig. 1) developing at a great distance from the nozzle aperture, the system of equations (15-16) reduces to the equations for a field of constant density. Geometrical and mechanical similarity will be assumed for the mixing field in cross sections at right angles to the jet axis.

As an example of the theoretical treatment [9], we describe the case of a round hot-air jet in stationary surrounding air:

$$u_1 = 0.$$

By dimensional analysis, we obtain:

the similarity coordinate is

$$\eta = \frac{r/r_0}{x/r_0}.$$

the width of the velocity field is set equal to

$$b_1/r_0 = B\, x/r_0,$$

the velocity distributions are

$$\frac{\bar{u}}{u_0} = \frac{U}{x/r_0}\, \varphi(\eta), \qquad \frac{\bar{v}}{u_0} = \frac{1}{x/r_0}\, \psi(\eta)$$

and the temperature distribution is

$$\vartheta/\vartheta_0 = \frac{\Theta}{x/r_0}\, \chi(\eta) \qquad (30)$$

with parameters B, U, Θ.

One then obtains the following equations:

continuity:

$$\psi + \eta \frac{d\psi}{d\eta} = U\, \eta \left(\varphi + \eta \frac{d\varphi}{d\eta}\right),$$

momentum:

$$-U\left(\varphi^2 + \eta\, \varphi \frac{d\varphi}{d\eta}\right) + \psi \frac{d\varphi}{d\eta} = \varkappa_1\, B\, U \left(\frac{d^2\varphi}{d\eta^2} + \frac{1}{\eta} \frac{d\varphi}{d\eta}\right), \qquad (31)$$

energy:

$$-U\, \varphi \left(\chi + \eta \frac{d\chi}{d\eta}\right) + \psi \frac{d\chi}{d\eta} = E\, \varkappa_1\, B\, U \left(\frac{d^2\chi}{d\eta^2} + \frac{1}{\eta} \frac{d\chi}{d\eta}\right).$$

The boundary conditions are

$$\left.\begin{array}{ll}\varphi(0) = \chi(0) = 1\,, & \\ \varphi(\pm\infty) = \chi(\pm\infty) = 0\,, & \psi(0) = 0\,;\end{array}\right\} \qquad (32)$$

The resulting solutions are /17

$$\varphi = \frac{1}{[1 + (\sigma\,\eta)^2]^2}\,, \qquad \sigma\,\psi = \frac{U}{2}\,\sigma\,\psi\,\frac{1 - (\sigma\,\eta)^2}{[1 + (\sigma\,\eta^2)]^2}$$

with the expansion factor

$$\sigma = \frac{1}{\sqrt{8\,\varkappa_1\,B}}\,; \qquad \chi = \varphi^{1/B}\,. \qquad (33)^7$$

The parameters U and Θ determining the axial functions

$$\frac{\bar{u}_A(x)}{u_0} = \frac{U}{x/r_0}\,, \qquad \frac{\bar{\vartheta}_A(x)}{\vartheta_0} = \frac{\Theta}{x/r_0}$$

-- the first function describes the velocity drop along the jet axis, and the second the temperature drop along it -- and the parameter B defined by $b_1/r_0 = B\,x/r_0$ are determined by means of energy and momentum conservation, expressing the constant momentum and energy flux through planes perpendicular to the jet axis:

the momentum integral

$$\int_0^\infty \varrho\,u^2\,r\,dr = \varrho_0\,u_0^2\,\frac{r_0^2}{2} \qquad (34)$$

(the index 0 denotes the nozzle aperture)
yields, when the functions in (33) are substituted, the relation

$$\sigma^2 = \frac{U^2}{3}\left(1 + \frac{\vartheta_0}{T_1}\right). \qquad (35)$$

The energy integral

$$\int_0^\infty \varrho\,u\,\vartheta\,r\,dr = \varrho_0\,u_0\,\vartheta_0\,\frac{r_0^2}{2} \qquad (36)$$

furnishes the relation

[7] The formulas for the velocity field have already been given in Grenzschichttheorie [Boundary layer theory] by H. Schlichting, (Karlsruhe, 1951).

$$\sigma^2 = \frac{U\,\Theta}{1+(2/E)}\left(1+\frac{\vartheta_0}{T_1}\right). \tag{37}$$

By reading off the width σB of the calculated profile ϕ, and using the above relations, the following formulas are obtained for the expansion factor σ and the parameters

$$\left.\begin{aligned} \sigma &= \frac{1}{8\,\varkappa_1(\sigma B)}, \\ U &= \frac{\sqrt{3}}{8\,\varkappa_1(\sigma B)}\,\frac{1}{[1+(\vartheta_0/T_1)]^{1/2}}, \qquad \Theta = \frac{1+(2/E)}{3}\,U, \\ B &= 8\,\varkappa_1(\sigma B)^2. \end{aligned}\right\} \tag{38}$$

One thing which can be seen from the formulas is that the hotter the jet is, the steeper the drop in velocity and temperature along the jet axis. Since the transfer ratio $E = 2$ (Eq. 1), the temperature drops faster than the velocity along the jet axis.

The measurements depicted in Figs. 10 and 11 for the axial functions at temperatures 15°, 170°, and 300° above that of the outside air can be considered confirmation of the theory.

Now reading off the width σB of the calculated profile /19 ϕ -- as in Section 1 -- between the limits 0.95 and 0.05, comparison with the measurements in the asymptotic region in stationary outside air yields $x_1 = 0.0085$ for the empirical coefficient and $x_1 = 0.0082$ when the surrounding air is in motion. The measurements in the core region (Section 1) supplied $x_1 = 0.0082$. The empirical coefficient x_1 can accordingly be considered essentially constant over the entire region of turbulent expansion in a round jet from the nozzle aperture to the asymptote.

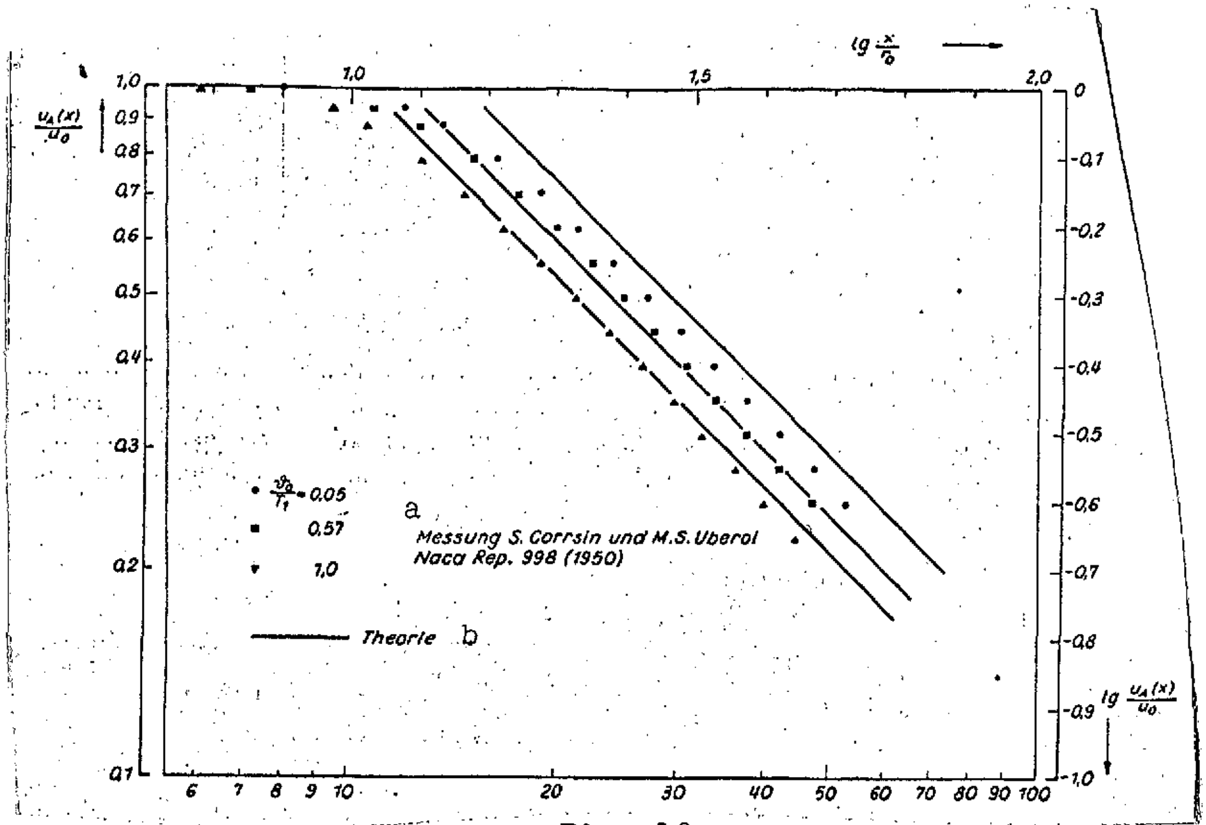

Fig. 10.

Key: a. Measurement b. Theory

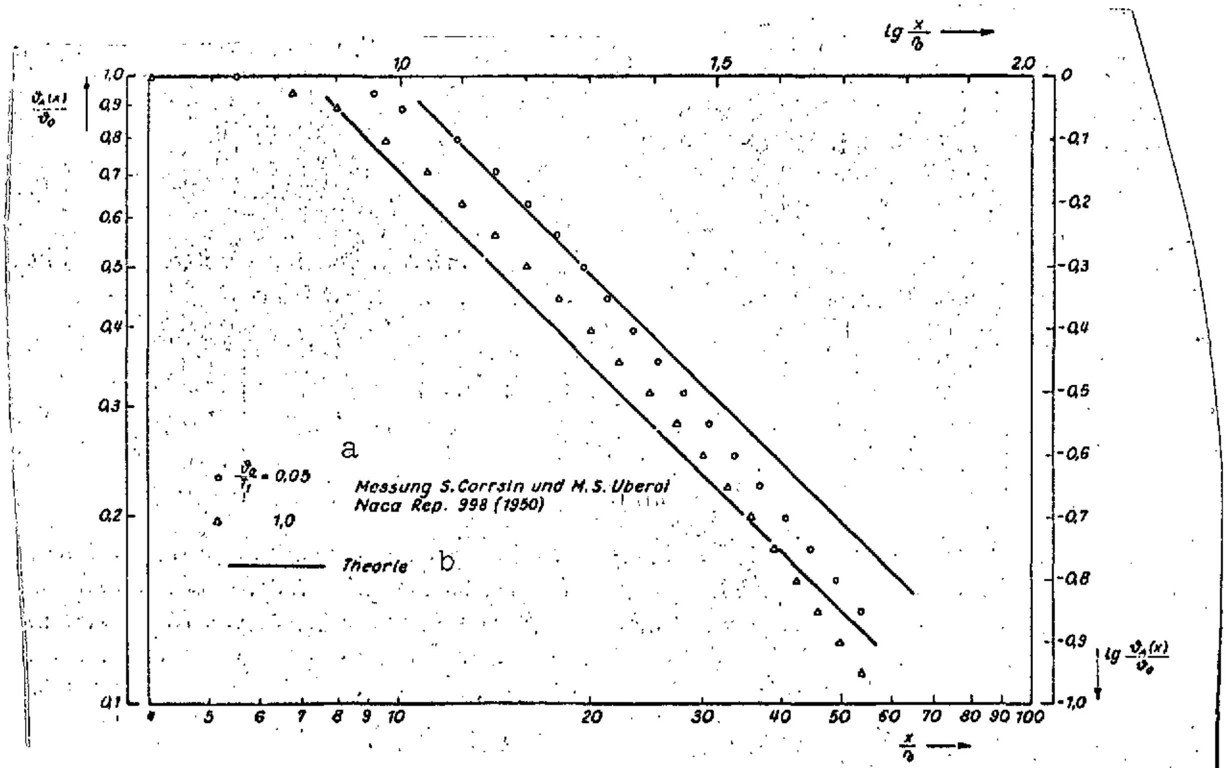

Fig. 11.

Key: a. Measurement b. Theory

III. Energy Conservation for Compressible Turbulent Boundary Layers

The flow processes in jet engines and in boundary layers adhering to objects in flight are known to differ from exterior flow, which can be considered as potential flow, in that molecular and turbulent effects such as friction, heat conduction, and dissipation, become crucial in the boundary layer. This alone is sufficient to justify the great technical importance of boundary-layer research in the theory of jet engines of all types.

When flight velocities are large (supersonic and hypersonic velocities), the wall is heated much more intensely than in exterior flow, because of compression and particularly dissipation. Since, according to the ideas of boundary-layer theory, the pressure in cross sections perpendicular to the wall can be viewed as constant, the state equation of ideal gases at constant pressure

$$\rho T = \text{const},$$

implies that the variation in temperature over the thickness of the boundary layer causes the density to vary over the same region. Hence, in the case of turbulence, we are dealing with a compressible turbulent boundary layer.

The free-turbulence flows discussed in Section II differ fundamentally from the turbulent boundary-layer flows in that molecular effects in the latter are not negligible. As far as the transport processes (friction, heat conduction) are concerned, molecular transport is no longer negligible in comparison to turbulent transport in a thin zone of the boundary layer adjacent to the wall, since here the wall suppresses the macroscopic turbulent mixing process; this is the so-called laminar sublayer of turbulent boundary layers, which, including a transition region, is only about 1% as thick as the boundary layer itself. Lastly, with large relative air speeds, dissipation in the entire boundary layer is a crucial term in the energy balance.

We are still a long way from a theory for turbulent boundary layers, even one of merely phenomenological character such as the theory of free turbulence presented in Section II. Theoretical statements on the mean state parameters are available in essence only for a fully turbulent partial layer (i.e. with negligible molecular transport) adjacent to the laminar sublayer, the partial layer being characterized by the negligibility of the convection of the flow and of the external forces, and hence of the pressure gradients of the external incompressible fluid flow which direct the boundary-layer flow. Neglecting both terms is legitimate because of the nearness of this layer to the wall: the flow clings to the wall, which makes convection insignificant in this case; also, the pressure gradient -- unless we are very near separation of the boundary layer -- is negligible in comparison with the large values assumed for the tangential stress forces. The layer characterized in this way is admittedly relatively thin; however, because of the very steep velocity and temperature gradients close to the wall, it includes a relatively /20
large proportion of the total velocity rise or temperature drop across the boundary layer. This gives this layer its importance. /20

Consistently allowing for the density fluctuations occurring in the turbulent boundary layer, we obtain a version of the energy-conservation equation for this layer which is different from previous formulations[8]. Density fluctuations will inevitably occur in compressible turbulent boundary layers because of the density distributions existing in these layers. This is immediately evident, since the process of turbulent mixing can be thought of

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[8] Cf. e.g. E.R. van Driest [Jounr. Aeron. Sci. 18, 145 (1951)] and J. Rotta [Z. Flugwiss. 7, 204 (1959)].

as a consequence of the migration of whole fluid particles from one layer to the next. Momentum and energy conservation then furnish a new law for the relation between temperature and velocity in this layer. We observe that this relation plays a fundamental role in practical computation procedures for compressible turbulent boundary layers.

In accordance with the ideas of boundary-layer theory, the computation allows for laminar transport due to molecules (which, for the time being, we continue to take into account), turbulent transport due to fluid elements, and their gradients only in the direction at right angles to the wall.

We have already formulated in Section II the main principle for determining the flow-process correlations crucial for turbulent transport.

For the time being, we can retain the continuity equation (6) unchanged.

continuity:

$$\frac{\partial \bar{\varrho}\,\bar{u}}{\partial x} + \frac{\partial \bar{\varrho}\,\bar{v}}{\partial y} = \partial/\partial y\left(-\overline{\varrho' v'}\right). \tag{39}$$

If we neglect the convection terms in the momentum equation (7), but still allow for molecular momentum transport, we obtain by integrating over y

momentum:

$$\tau_0 = \overline{\mu \frac{\partial u}{\partial y}} + \bar{\varrho}\left(-\overline{u' v'}\right) + \bar{u}\left(-\overline{\varrho' v'}\right), \tag{40}$$

where $\overline{\mu(\partial u/\partial y)}$ with the viscosity coefficient μ represents the mean laminar shear stress and τ_0 the mean wall shear stress (here too, the index 0 denotes the mean values at the wall). Equation (40) obviously describes a layer of constant shear stress.

In complete generality, the energy equation reads

$$\operatorname{div}\left[\overline{\left(c_p T + \frac{u^2+v^2+w^2}{2}\right)\varrho\,\mathfrak{v}}\right] = \frac{\partial(\overline{\lambda \operatorname{grad} T})}{\partial y} + \partial/\partial y\left\{\overline{\mu\, u\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\mu\, v\left(2\frac{\partial v}{\partial y}-\frac{2}{3}\operatorname{div}\mathfrak{v}\right)+\mu\, w\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)}\right\}. \qquad (41)$$

Here z is the coordinate perpendicular to the x-y plane and w is the corresponding velocity component; λ is the coefficient of thermal conduction.

We first observe that for boundary layers

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$$\overline{\frac{u^2+v^2+w^2}{2}\varrho\,\mathfrak{v}} \approx \frac{1}{2}\,\overline{u^2\varrho\,\mathfrak{v}}.$$

By the main principle, we again obtain

$$\overline{u^2\varrho\, v} \approx \bar{u}^2\,\bar{\varrho}\,\bar{v} + 2\,\bar{\varrho}\,\bar{u}\,\overline{u'v'} + \bar{u}^2\,\overline{\varrho' v'},$$
$$\overline{c_p T\varrho\, v} \approx c_p\left(\bar{\varrho}\,\bar{T}\,\bar{v} + \bar{\varrho}\,\overline{T'v'} + \bar{T}\,\overline{\varrho' v'}\right).$$

Consequently /21

$$\operatorname{div}\left[\left(c_p\bar{T}+\frac{\bar{u}^2+\bar{v}^2+\bar{w}^2}{2}\right)\bar{\varrho}\,\bar{\mathfrak{v}}\right] = \partial/\partial y\left(\overline{\lambda\frac{\partial T}{\partial y}}\right) + \partial/\partial y\left\langle\bar{\varrho}\left(-c_p\overline{T'v'}\right)+c_p\bar{T}\left(-\overline{\varrho' v'}\right)\right\rangle + \partial/\partial y\{\overline{}\} + \partial/\partial y\left\langle\bar{\varrho}\,\bar{u}\left(-\overline{u'v'}\right)+\frac{\bar{u}^2}{2}\left(-\overline{\varrho' v'}\right)\right\rangle. \qquad (42)$$

The terms in the angular brackets on the right side represent the turbulent energy transport.

Keeping the continuity equation (39) in mind, we can replace T in (42) by $\theta = T - T_0$. If the convection terms are again neglected in this layer, integration over y yields the equation

$$\lambda_0\left(\frac{\partial\bar{\vartheta}}{\partial y}\right)_0 = \overline{\lambda\frac{\partial\vartheta}{\partial y}} + \bar{\varrho}\left(-c_p\overline{\vartheta' v'}\right) + c_p\bar{\vartheta}\left(-\overline{\varrho' v'}\right) + \{\overline{}\} + \bar{\varrho}\,\bar{u}\left(-\overline{u'v'}\right) + \frac{\bar{u}^2}{2}\left(-\overline{\varrho' v'}\right). \qquad (43)$$

Accordingly, this is a layer of constant energy transport.

When restricting matters to the completely turbulent layer, we can ignore the molecular transport terms in (40) and (43) in comparison with the turbulent ones. Then, using (40), we finally obtain from (43) the following new form of the energy equation

$$\lambda_0 \left(\frac{\partial \bar{\vartheta}}{\partial y}\right)_0 = \bar{\varrho}\left(-c_p \overline{\vartheta' v'}\right) + c_p \bar{\vartheta}\left(-\overline{\varrho' v'}\right) + \underline{\tau_0 \bar{u} - \frac{\bar{u}^2}{2}\left(-\overline{\varrho' v'}\right)}. \qquad (44)$$

The two underlined terms can be thought of as the dissipation (frictional heat) of the flow. The noteworthy result is that in addition to the term $\tau_0 \bar{u}$, which represents both the dissipation of the principal motion and the energy drawn from the latter to preserve the perturbation motion, there is also a term $\bar{u}^2 \overline{\rho' v'}/2$, which can be viewed as the diffusion of kinetic energy by means of turbulent mass transport.

As a consequence, which we will test by experiment, we derive from (40) and (44) a new relationship between temperature and velocity for this layer.

In order to use Eqs. (40) and (44) to obtain information on the mean state parameters, we first need a hypothesis (as in Section II) linking the fluctuation correlations appearing in the equations with the mean state parameters. For our purposes, it is sufficient to employ the very general approach of Boussinesq:

$$\left.\begin{aligned} -\overline{u' v'} &= \varepsilon(x, y)\frac{\partial \bar{u}}{\partial y}, \\ -\overline{\vartheta' v'} &= E\, \varepsilon(x, y)\frac{\partial \bar{\vartheta}}{\partial y}, \\ -\overline{\varrho' v'} &= E\, \varepsilon(x, y)\frac{\partial \bar{\varrho}}{\partial y}. \end{aligned}\right\} \qquad (45)$$

In these equations, $\varepsilon(x,y)$ has the dimensions of a kinematic viscosity. The coefficient E represents the so-called transfer ratio (cf. Section II).

While the first two trial solutions are common approaches in the theory of turbulent boundary layers, the following should be mentioned about the last equation: by analogy with kinetic gas theory, the above equations represent the idea that fluid properties are transported from one layer to the next. The first two equations express this for velocity and temperature. Since, in a compressible boundary layer, different layers will also have different densities, we must also assume a corresponding transport of density from one layer to the next. The last /22
equation is just the appropriate formulation.

We thus obtain:

momentum:

$$\tau_0 = \varepsilon\left(\bar{\varrho}\frac{\partial \bar{u}}{\partial y} + E\,\bar{u}\frac{\partial \bar{\varrho}}{\partial y}\right), \tag{46}$$

energy:

$$\lambda_0\left(\frac{\partial \bar{\vartheta}}{\partial y}\right)_0 = E\,\varepsilon\, c_p \frac{\partial \bar{\varrho}\,\bar{\vartheta}}{\partial y} + \tau_0\,\bar{u} - E\,\varepsilon\,\frac{\bar{u}^2}{2}\frac{\partial \bar{\varrho}}{\partial y}. \tag{47}$$

By using the state equation for ideal gases at constant pressure

$$\bar{\varrho}\,\bar{T} \approx \varrho_0\, T_0$$

we can replace $\bar{\rho}$ by $\bar{T} = T_0 + \bar{\theta}$.

Now introducing the so-called shear-stress velocity $v_* = \sqrt{\tau_0/\rho_0}$, the dimensionless coordinates

$$\eta = \frac{v_*\, y}{\nu_0}, \quad \omega = \frac{\bar{u}}{v_*} \quad \text{and} \quad \chi = \bar{\vartheta}/T_0 \quad \left(\nu_0 = \frac{\mu_0}{\varrho_0}\right)$$

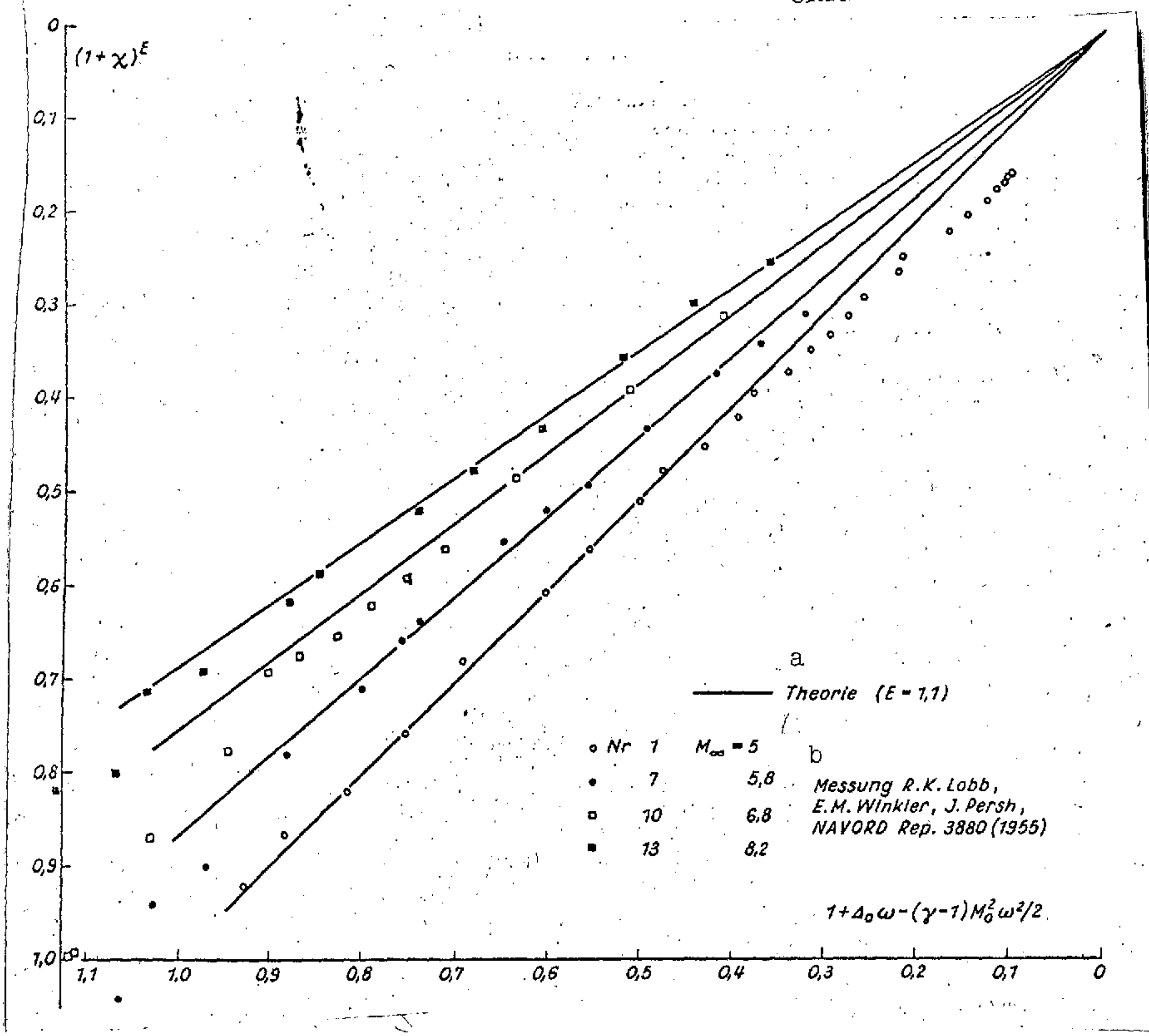

Fig. 12.

Key: a. Theory
b. Measurement

and the parameters

$$\Lambda_0 = \frac{\lambda_0 \left(\frac{\partial \vartheta}{\partial y}\right)_0}{c_p \varrho_0 T_0} \cdot \frac{1}{v_*}, \qquad M_0 = \frac{v_*}{\sqrt{(\gamma-1)c_p T_0}}, \qquad (\gamma = c_p/c_v)$$

we obtain

momentum:

$$1 = \frac{\varepsilon}{\nu_0}\left\{\frac{1}{1+\chi}\frac{\partial\omega}{\partial\eta} - E\,\omega\,\frac{\partial\chi/\partial\eta}{(1+\chi)^2}\right\}, \tag{48}$$

energy:

$$A_0 - (\gamma-1)\,M_0^2\,\omega = E\,\frac{\varepsilon}{\nu_0}\,\frac{\partial\chi/\partial\eta}{(1+\chi)^2}\left\{1 + (\gamma-1)\,M_0^2\,\frac{\omega^2}{2}\right\}. \tag{49}$$

These two equations furnish the new relation between temperature and velocity

$$(1+\chi)^E = C\left[1 + A_0\,\omega - (\gamma-1)\,M_0^2\,\frac{\omega^2}{2}\right] \tag{50}$$

with the integration constant C.

We can also write

$$E \lg(1+\chi) = \lg\left[1 + A_0\,\omega - (\gamma-1)\,M_0^2\,\frac{\omega^2}{2}\right] + \lg C.$$

(By comparison, older theories, which ignore the density fluctuations, provide the relation

$$E\,(1+\chi) = \left[1 + A_0\,\omega - (\gamma-1)\,M_0^2\,\frac{\omega^2}{2}\right] + \mathrm{const}).$$

To compare this with an experiment, we used an American measurement [11] in the hypersonic range. The measurement supplied

$$E = 1.1 \tag{51}$$

which is consistent with the findings of other authors. From /23 this comparison, we exhibit Fig. 12, in which the given Mach numbers M_∞ refer to the Mach numbers of the flow outside the boundary layer. In the graphic representation of Fig. 12, relation (50) is depicted by a collection of rays with the zero point as the origin. This comparison can be taken as confirming the new relation.

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